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Subleading Contributions from Instanton Corrections in N = 2 Supersymmetric Black Hole Entropy

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Abstract

We present subleading corrections to the $N = 2$ supersymmetric black hole entropy. These subleading contributions correspond to instanton corrections of the Type II string theory. In particular we consider an axion free black hole solution of low-energy effective Type II string theory. We present a procedure to include successiv all instanton corrections. Expanding these corrections at particular points in moduli space yields polynomial and logarithmic instanton corrections to the classical black hole entropy. We comment on a microscopic interpretation of these instanton corrections and find that the logarithmic corrections correspond to subleading terms in the degeneracy of the spectrum of an underlying quantum theory.

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1. Introduction

In recent times there has been substantial progress towards a better understanding of the microscopic origin of the Bekenstein-Hawking entropy. Using the D-brane approach a microscopic origin for certain black hole solutions was found [1]. Using the fact that these black holes are compactified string solution one can explore the known degeneracy formula of string states to find a statistical interpretation of the Bekenstein-Hawking entropy. In this context Susskind speculated that the Bekenstein-Hawking entropy might be explained in terms of the degeneracy of free strings [2]. For large level N this degeneracy reads

$$d(N) = e^{S_{stat}} = N^{-\frac{c+3}{4}} \; e^{2\pi\sqrt{\frac{c}{6}N}} \tag{1}$$

where c is the central charge. The exponential term is the leading term while the polynomial part is known as the subleading term. This picture has been recently reinvestigated and expanded in [3].

All supersymmetric black hole solutions with finite horizon in four dimensions have a natural embedding as BPS states, breaking one half of $N = 2$ supersymmetry. For all these solutions the leading exponential term in (1) could be identified at the classical level (torus compactification) [4] and if one includes quantum corrections [5]. On the other hand the origin of the subleading polynomial terms in the context of the Bekenstein-Hawking entropy of black hole solutions has up to now no analogous explanation. The main purpose of the present paper is to fill this gap. We will restrict ourselves to the entropy of BPS saturated Type II black hole solutions. In a forthcoming paper we will discuss the corresponding black hole solutions of effective Type II and heterotic string theory and the corresponding role of the subleading corrections in the context of heterotic/Type II duality symmetries [6]. For Schwarzschild black holes the role of the subleading corrections has recently been addressed in [7]..

BPS saturated black hole solutions become independent of the moduli on the horizon [8]. This result can be understood from a supersymmetric point of view [9]: The central charge appearing in the supersymmetry algebra reaches an extremum in moduli space. The corresponding extremization problem is equivalent to the solution of the equations (p and q are the magnetic and electric charges)

$$Y^I - \bar{Y}^I = ip^I \qquad , \qquad F_I - \bar{F}_I = iq_I \tag{2}$$

with the symplectic vector

$$V = \begin{pmatrix} Y^I \\ F_I \end{pmatrix} \tag{3}$$

which defines the special geometry of the theory ($F_I = \partial F/\partial Y^I$ and F is the prepotential) and the index $I = 0, 1, ..n$ where n is the number of vector multiplets. Note that in this basis the symplectic constraint is: $i\langle\bar{V}, V\rangle = |Z|^2$, where Z is the $N = 2$ central charge (for details see [5]). The equations (2) determine uniquely the $n+1$ fields Y^I in terms of the charges and

parameters of the prepotential, like the intersection numbers, Euler and instanton numbers. Taking this solution the Bekenstein-Hawking entropy is

$$S = i\pi \, \left(\bar{Y}^I F_I(Y) - Y^I \bar{F}_I(\bar{Y})\right)_{|extr.} \; = \; \pi \, |Z|^2_{|extr.} \tag{4}$$

2. The Type II Solution

We define the Kahler class moduli t^A by $t^A = Y^A/Y^0$ and the prepotential reads [10]

$$F(Y^0, Y^A) = (Y^0)^2 \left[-\frac{1}{6} C_{ABC} t^A t^B t^C - \frac{i\chi\zeta(3)}{2(2\pi)^3} + \frac{i}{(2\pi)^3} \sum_{d_1,..d_n} n^r_{d_1,..,d_n} Li_3(e^{2\pi i \, d_A t^A}) \right] . \tag{5}$$

Here C_{ABC} are the classical intersection numbers, χ is the Euler number, $n^r_{d_1,..,d_n}$ are the numbers of genus zero rational curves (instanton numbers) and d_A are the degrees of these curves. These numbers are determined by the Calabi-Yau threefold of the compactified effective Type II string theory.

Restricting ourselves to the axion free case ($t^A + \bar{t}^A = 0$) with $p^0 = q_A = 0$ we find as a solution of (2)

$$Y^0 = \frac{\lambda}{2} \quad , \quad Y^A = i\,\frac{p^A}{2} \qquad \text{and} \qquad t^A = i\,\frac{p^A}{\lambda} \tag{6}$$

and the parameter λ is fixed by the constraint $F_0 - \bar{F}_0 = iq_0$. Inserting the solution into (5) this constraint becomes

$$4\frac{\partial}{\partial\lambda} F(\lambda, p^A) = iq_0 \; . \tag{7}$$

Note that for the axion free solution F_0 is pure imaginary and F_A is real. Inserting the fields in the entropy (4) we obtain

$$S = \frac{\pi}{2} \left(-\lambda q_0 + \frac{1}{2\lambda} C_{ABC} p^A p^B p^C - \frac{\lambda}{(2\pi)^2} \sum_{d_1,..d_n} n^r_{d_1,..,d_n} \, d_A p^A \; Li_2(e^{-2\pi d_A p^A/\lambda}) \right) \; . \tag{8}$$

With a given solution for λ the entropy includes all instanton corrections. However, to find a well-defined entropy ($S \geq 0$) one has to adjust the signs of the charges in an appropriate way. At the classical level and for the example we discuss below the entropy is well-defined for large negative electric charge q_0.

The entropy formula (8) is implicit and it seems difficult to find a general analytic solution. However, we can expand the solution at special points in moduli space. Note first of all that for large moduli ($t^A \gg 1$) the instanton part is exponentially suppressed and the solution is determined by the pure cubic part. Due to the instanton nature it is impossible to expand around this point, since we cannot obtain these terms in perturbation theory. On the other hand, if at least one of the moduli is small, say t^3, one finds a non-trivial expansion

of the instanton corrections near this special point in moduli space. Thus, we consider the region

$$p^3 \ll \lambda \ll p^A \tag{9}$$

where $A \neq 3$. This limit corresponds to the case where one 4-cycle is small, which is related to t^3, and all other 4-cycles are large. In this case the instanton correction reads

$$\sum_{d_A} n^r_{d_A} Li_3(e^{-2\pi d_A p^A/\lambda}) = \sum_{d_3} n^r_{d_3} \left(\zeta(3) - \frac{(2\pi)^3 d_3 p^3}{24\,\lambda} + (\frac{3}{4} - \frac{1}{2}\log\frac{2\pi d_3 p^3}{\lambda})(\frac{2\pi d_3 p^3}{\lambda})^2 \pm .. \right) , \tag{10}$$

with $n^r_{d_A} \equiv n^r_{d_1,..,d_n}$ and $n^r_{d_3} = n^r_{0,0,d_3,0,..}$. The logarithm corresponds to the non-analytic part of the trilogarithm. If we include further terms the structure remains

$$Li_3(e^{-x}) \approx p(x) + q(x)\log x \tag{11}$$

where $p(x)$ and $q(x)$ are analytic polynomials for small x. Using this expansion we find for the constraint (7)

$$q_0\lambda^2 = -\frac{1}{6}C_{ABC}p^A p^B p^C + \frac{a\,\zeta(3)}{2(2\pi)^3}\lambda^3 - \sum_{d_3} n^r_{d_3}\left[\frac{d_3 p^3}{24}\lambda^2 - \frac{(d_3p^3)^2}{4\pi}\lambda\right] + \mathcal{O}((p^3)^3), \tag{12}$$

where the constant a collects all terms proportional to $\zeta(3)$. There is however a sublety in this expansion. Because there is no restriction on the degrees of the rational curves the sum over all instanton numbers ($n^r_{0,0,d_3,0,..}$) can be "in principle" infinite. However, calculations for the Calabi-Yau that is dual to the heterotic STU model indicate that this sum is finite ($n^r_{0,0,1} = -2$, $n^r_{0,0,d_3} = 0$ for $d_3 = 2,..,10$). This is crucial for the correct mapping of both theories in our approximation, see [13] and the second ref. of [10]. Therefore, we will assume that the sum converges. From (9) and (12) follows, that $q_0 \gg p^A$ and we can expand the solution as

$$\lambda = \sum_{i=1}\frac{\alpha_i}{(\sqrt{q_0})^i} = \pm\sqrt{\frac{-\frac{1}{6}C_{ABC}p^Ap^Bp^C}{q_0}} + \sum_{d_3} n^r_{d_3}\frac{(d_3p^3)^2}{8\pi q_0} \pm .. \,. \tag{13}$$

In the end we considered two expansions. First we used $p^3 \ll \lambda$ to obtain a power expansion in p^3. Then we expanded the solution of (7) in $1/\sqrt{q_0}$. Finally, in order to obtain the instanton corrections to the entropy, we have to expand (8) in the same way and insert this solution for λ. As a result the entropy contains a polynomial part and logarithmic terms. Taking only the leading terms for both parts into account (with the minus sign for λ) the entropy reads

$$S = 2\pi\sqrt{-q_0\frac{1}{6}C_{ABC}p^Ap^Bp^C} - \frac{1}{8}\sum_{d_3} n^r_{d_3}(d_3p^3)^2\log\left(\frac{(d_3p^3)^2 q_0}{-\frac{1}{6}C_{ABC}p^Ap^Bp^C}\right) \pm .. \tag{14}$$

or written in the exponential form

$$d = e^S = e^{S_0}\prod_{d_3}\left(\frac{(d_3p^3)^2 q_0}{-\frac{1}{6}C_{ABC}p^Ap^Bp^C}\right)^{-n^r_{d_3}(d_3p^3)^2/8} \tag{15}$$

with the classical entropy

$$S_0 = 2\pi\sqrt{-q_0\frac{1}{6}C_{ABC}p^Ap^Bp^C}. \tag{16}$$

Choosing a special Calabi-Yau space this solution can be "mapped on the heterotic side" [12] and has an analogous interpretation in a quantum corrected heterotic $N = 2$ model [11]. For the STU model our expansion corresponds to the heterotic prepotential

$$F = i(Y^0)^2\left[STU\ -\frac{1}{\pi}(T-U)^2\log(T-U)\ +\ \Delta(T,U)\right]\ . \tag{17}$$

Here the logarithmic and the polynomial corrections $\Delta(T,U)$ correspond to one-loop corrections near the line $T = U \neq 1$ [11]. If we take only the classical prepotential and the logarithmic quantum corrections into account, the entropy of the corresponding quantum corrected axion free black hole in the STU model with $N = |q_0|$ is

$$S = \pi\ \left(\sqrt{Np^1p^2p^3} - \frac{1}{2\pi}(p^2-p^3)^2\ \log\sqrt{N}\right). \tag{18}$$

Note that p^1 is an electric charge in the STU model.

3. The Microscopic Picture

Up to now we have considered an instanton corrected black hole solution of an effective string field theory in four dimensions. However, the ultimate goal is to find a statistical interpretation of the black hole entropy in terms of a degeneracy of states of an underlying quantum theory. At the classical level - without instanton corrections - this has been established for the model under consideration for the extremal and near-extremal case in [14]. In this case the entropy is given by S_0 (16) only. The main microscopic interpretation is as follows: The black hole carries one electric and n magnetic charges, corresponding to an electric graviphoton and n magnetic vector gauge fields. There are two different ways to consider the solution. First, as a compactification of the Type II string theory or, second, as a compactification of the equivalent M-brane configuration. The latter one is of special interest. In this case one first compactifies the configuration down to five dimensions on a Calabi-Yau manifold. In this case the 11d configuration is an intersection of three 5-branes. These 5-branes have to intesect over a common string [15]. After compactification, this yields a magnetic string solution in 5d. Then one wraps this magnetic string around the $5th$ direction. Thus, the Bekenstein-Hawking entropy of the black hole is directly related to the statistical entropy of a string. The corresponding black hole states are momentum states of this magnetic string giving rise to the electric charge in four dimensions. In this procedure one uses the known formula for the degeneracy of string states, but one should keep in mind, that one does not count states of the closed Type II string. The momentum modes in 5d correspond to open membranes in 11d which are attached to the M-5-branes. All these states are located at the intersection of the 5-branes, since they become massless there. In a Calabi-Yau manifold the number of these intersections is given by C_{ABC}. Furthermore, every intersection consists of

many layers of 5-branes, which means that we have wrapped each of the 5-brane many times around the different 4-cycles. This multiple wrapping is counted by the magnetic charge p^A. Summing up all these contribution one finds that the Bekenstein-Hawking entropy (16) coincides in the leading order (exponential term) with the statistical entropy of string states

$$d(N) = e^{S_{stat}} \sim N^{-\frac{r}{4}}\, e^{2\pi\sqrt{\frac{c}{6}N}} \ . \tag{19}$$

Here $N = -q_0$ is the oscillator number level and r is a parameter. For a free string one has $r = c + 3$ with $c = C_{ABC}p^A p^B p^C$ as the effective central charge of the 5d string.

Our entropy formula (15) indicate, that at least in parts the polynomial term of this statistical entropy (19) can be interpreted as instanton corrections. What is the microscopic picture of these terms? The instanton terms correspond to mapped (euclidean) closed string world sheets into the Calabi-Yau. This mapping might be of higher degree, e.g.. multiple wrapping counted by the numbers d_A. Furthermore, the instanton numbers $n^r_{d_1,\ldots,d_n}$ count the number of rational curves upon which we map the string worldsheet. However, in M-theory we do not have string worldsheets which we could wrap in the Calabi-Yau. On the other hand one can find the following interpretation in M-theory: Analogous to the string worldsheet we have to wrap membranes completely into the internal space. From the three (euclidean) worldvolume coordinates, one is the $5th$ direction and the others are mapped onto the rational curves. These are additional winding states (in addition to the momentum states discussed above). Thus, in 11d our configuration consists of i) the intersection of three 5-branes and ii) a gas of free closed membranes. On the Type II side the black hole is a compactified intersection of three 4-branes and one 0-brane. This configuration lives in a gas of free closed strings, which are mapped on rational curves upon compactification. This string gas is the origin of the subleading instanton corrections in the black hole entropy.

However, the occurrence of $\zeta(3)$ in (12) is a serious problem. In our approximation this term has been omitted. In contrast to all other parameters $\zeta(3)$ is not an integer or rational and cannot be expressed in terms of π. As consequence it seems to be difficult to find an appropriate microscopic interpretation. An interesting observation is that the statistical entropy of a massless ideal gas for a membrane is proportional to $\zeta(3)$ [16]: $S_{stat.} = 7/8\pi\, \zeta(3)\, N\, L^2\, T^2$ (N number of free bosons, L^2 is the spacial volume and T is the tension). It would be very interesting to find a connection between this statistical entropy and the $\zeta(3)$ correction to our Bekenstein-Hawking entropy, which are caused by wrapped membranes. These $\zeta(3)$ terms can be separated by considering a region where all magnetic charges (which are related to 5-branes) are neglected. For this pure instantonic case we obtain

$$S = \frac{(2\pi)^4 (q_0)^2}{2|a|\, \zeta(3)} \ . \tag{20}$$

Turning off all 5-branes means that we have neglected all vector multiplet contributions to the entropy except the one of the $N = 2$ supergravity multiplet, which contains the graviphoton. Thus, this contribution has its origin in pure $N = 2$ supergravity.

4. Summary

In this paper we have addressed the question of subleading contributions to the entropy of $N = 2$ supersymmetric black holes. In low-energy effective Type II theories in four dimensions these corrections have a possible origin in instanton corrections. We have chosen particular points in moduli space to expand the instanton corrections. The calculations base on an expansion around a vanishing 4-cycle in the Calabi-Yau space and the assumption that all other 4-cycles are large. In the context of the dual heterotic STU model this expansion corresponds to the region $T \simeq U$ in moduli space. Since the corresponding black hole is a compactified string solution in 5d, it is obvious to connect the Bekenstein-Hawking entropy with the statistical entropy of an underlying quantum theory. Since the formulas (1) and (14) have the same structure we find that parts of the full instanton correction can serve as subleading terms in the degeneracy formula of the underlying quantum theory. They are corrections in a $1/\sqrt{N}$ ($N = |q_0|$) expansion around a vanishing 4-cycle ($p^3 \simeq 0$).

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